Handling spatial dependence under unknown unit locations

Supplementary material

1 Proof of Equation (7)

Proof. According to LeSage and Pace (2009, p. 47), the log-likelihood function of the SLM (2) is:

$$\ln \mathcal{L}(\rho; W) = -\frac{n}{2} \ln(2\pi\sigma^2) + \ln \det(A) - \frac{1}{2\sigma^2} (Ay - X\beta)^{\mathrm{T}}(Ay - X\beta)\,, \quad \text{(SM.1)}$$

where $A = I_n - \rho W$, as defined in the paper.

If (SM.1) is concentrated with respect to ρ, we have (cf. LeSage and Pace 2009, p. 48):

$$\ln \mathcal{L}_c(\rho; W) = -\frac{n}{2} \ln \frac{2\pi \mathrm{e}}{n} + \ln \det(A) - \frac{n}{2} \ln \left[(MAy)^{\mathrm{T}}(MAy)\right]\,, \quad \text{(SM.2)}$$

where $M = I_n - X(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}$ is the projection matrix defined in the paper.

Analogously, the concentrated log-likelihood of model (5) is

$$\ln \mathcal{L}_c(\rho_X; W_X) = -\frac{n}{2} \ln \frac{2\pi \mathrm{e}}{n} + \ln \det(A_X) - \frac{n}{2} \ln \left[(MA_Xy)^{\mathrm{T}}(MA_Xy)\right]\,. \quad \text{(SM.3)}$$

Equation (7) is obtained by subtracting Equation (SM.2) from (SM.3). □

2 Proof of Equation (8)

Proof. In case of squared real matrices (as spatial weight matrices are), the Frobenius norm $\|\cdot\|_F$ is defined as $\|A\|_F = \sqrt{\mathrm{tr}(AA^{\mathrm{T}})}$, for some $A \in \mathbb{R}^{n\times n}$ (Horn and Johnson 2013, p. 341). Thus, properties of the trace of matrices imply that:

$$\begin{aligned}
\|\rho_X W_X - \rho W\|_F^2 &= \mathrm{tr}\left((\rho_X W_X - \rho W)(\rho_X W_X - \rho W)^{\mathrm{T}}\right) = \\
&= \mathrm{tr}\left((\rho_X W_X)(\rho_X W_X)^{\mathrm{T}}\right) + \mathrm{tr}\left((\rho W)(\rho W)^{\mathrm{T}}\right) - 2\rho_X \rho\, \mathrm{tr}(W_X W^{\mathrm{T}}) = \\
&= \|\rho_X W_X\|_F^2 + \|\rho W\|_F^2 - 2\rho_X \rho\, \mathrm{tr}(W W_X^{\mathrm{T}}) \\
&= \|\rho_X W_X\|_F^2 + \|\rho W\|_F^2 - 2\rho_X \rho\, \mathrm{tr}(W_X^{\mathrm{T}} W)\,.
\end{aligned}$$

Standard matrix algebra permits the following identity to be verified for any quadratic form:

$$x^{\mathrm{T}}Ax = \mathrm{tr}(Axx^{\mathrm{T}})\,.$$

Thus, the trace $\mathrm{tr}(W_X^{\mathrm{T}}W)$ can be seen as the expected value of the quadratic form $u^{\mathrm{T}}W_X^{\mathrm{T}}Wu$, where $u \in \mathbb{R}^n$ is a random vector such that $\mathbb{E}(u) = 0 \in \mathbb{R}^n$ and $\mathbb{E}(uu^{\mathrm{T}}) = I_n$. □

3 Proof of Equations (10) and (11)

Proof. In order to prove Equations (10) and (11), some preliminary results are first derived.

Since both $W_{(m)}$ and $W_{(\lfloor\alpha m\rfloor)}$ are binary matrices, we have that:

$$W_{(m)}\iota_n = m\,, \qquad\qquad W_{(\lfloor\alpha m\rfloor)}\iota_n = \lfloor\alpha m\rfloor\,, \tag{SM.4}$$

where $\iota_n = [1,\dots,1]^{\mathrm{T}} \in \mathbb{R}^n$.

The perturbation Equation (9) in matrix form becomes:

$$\tilde{W}_\alpha = A + [W_{(\lfloor\alpha m\rfloor)} - A] \odot B\,,$$

endequation where $\odot$ is the Hadamard matrix multiplication (that is, the elementwise multiplication). Just like in case of perturbation (9), $B_{ij} \sim \mathcal{B}(1-\gamma)$ for any $i \neq j$ and $B_{ii} = 0$ for any i, whereas the elements of A are distributed as $A_{ij} \sim \mathcal{B}(\lfloor\alpha m_i\rfloor/(n-1))$ if $i \neq j$, whilst $A_{ii} = 0$ for any i. Off-diagonal elements of A are statistically independent from off-diagonal elements of B.

Let define $Q_n \overset{\text{def}}{=} \iota_n\iota_n^{\mathrm{T}} - I_n \in \mathbb{R}^{n\times n}$. From the definition of A and B, it follows that:

$$\mathbb{E}(A) = (n-1)^{-1}\lfloor\alpha m\rfloor\iota_n^{\mathrm{T}} \odot Q_n\,, \qquad\qquad \mathbb{E}(B) = (1-\gamma)Q_n\,,$$

thus:

$$\begin{aligned}\mathbb{E}(\tilde{W}_\alpha) =& \mathbb{E}(A) + W_{(\lfloor\alpha m\rfloor)} \odot \mathbb{E}(B) - \mathbb{E}(A) \odot \mathbb{E}(B) = \\ =& (n-1)^{-1}\lfloor\alpha m\rfloor\iota_n^{\mathrm{T}} \odot Q_n + (1-\gamma)W_{(\lfloor\alpha m\rfloor)} + \\ & - (1-\gamma)(n-1)^{-1}\lfloor\alpha m\rfloor\iota_n^{\mathrm{T}} \odot Q_n = \\ =& \gamma(n-1)^{-1}\lfloor\alpha m\rfloor\iota_n^{\mathrm{T}} \odot Q_n + (1-\gamma)W_{(\lfloor\alpha m\rfloor)}\,,\end{aligned} \tag{SM.5}$$

since $Q_n \odot Q_n = Q_n$, and since for any spatial weight matrix W of order n, $W \odot Q_n = W$.

Since both $W_{(m)}$ and $W_{(\lfloor\alpha m\rfloor)}$ are binary matrices based on the nearest-neighbour criterion, we have that:

$$\begin{array}{rcll} (W_{(m)})_{ij} > 0 & \Rightarrow & (W_{(\lfloor\alpha m\rfloor)})_{ij} > 0 & \text{if } \alpha > 1 \\ (W_{(\lfloor\alpha m\rfloor)})_{ij} > 0 & \Leftrightarrow & (W_{(m)})_{ij} > 0 & \text{if } \alpha = 1 \\ (W_{(\lfloor\alpha m\rfloor)})_{ij} > 0 & \Rightarrow & (W_{(m)})_{ij} > 0 & \text{if } \alpha < 1 \end{array}$$

and thus:

$$W_{(m)} \odot W_{(\lfloor\alpha m\rfloor)} = W_{(\lfloor(1\wedge\alpha)m\rfloor)}\,. \tag{SM.6}$$

Moreover, note that:

$$\iota^{\mathrm{T}}m = n\bar{m}\,, \qquad\qquad m^{\mathrm{T}}m = n\bar{m}(1+\kappa_m^2)\,, \tag{SM.7}$$

by definition of $\bar{m} = \frac{1}{n}\sum_{i=1}^n m_i$ and $\kappa_m^2 = \frac{1}{n\bar{m}^2}\sum_{i=1}^n (m_i - \bar{m})^2$.

Finally, define the following quantities:

$$d_\alpha \overset{\text{def}}{=} \alpha m - \lfloor\alpha m\rfloor\,, \qquad\qquad \bar{d}_\alpha = n^{-1}\iota^{\mathrm{T}}d_\alpha\,, \tag{SM.8}$$

and note that, since $0 \leq d_\alpha < \iota_n$, the following inequality holds:

$$0 \leq \bar{d}_\alpha < 1 \qquad \text{for any } \alpha \in \mathbb{R}^+ \tag{SM.9a}$$

whereas

$$\bar{d}_\alpha = 0 \qquad \text{for any } \alpha \in \mathbb{N}. \tag{SM.9b}$$

Correlation between the terms of $\tilde{W}_\alpha u$ and $W_{(m)} u$ requires covariance and variances to be computed. The rest of the proof is focused on this.

The expected values of the elements of $\tilde{W}_\alpha u$ and $W_{(m)} u$ is zero, thus the covariance between them can be computed as follows:

$$\begin{aligned}
\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}(\tilde{W}_\alpha u)_i (W_{(m)} u)_i\right) &= \mathbb{E}\left(\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}(\tilde{W}_\alpha u)_i (W_{(m)} u)_i \,\middle|\, \tilde{W}_\alpha\right)\right) = \\
&= n^{-1}\,\mathbb{E}(\mathbb{E}(u^{\mathrm{T}} \tilde{W}_\alpha^{\mathrm{T}} W_{(m)} u | \tilde{W}_\alpha)) = \\
&= n^{-1}\,\mathbb{E}(\mathrm{tr}(\tilde{W}_\alpha^{\mathrm{T}} W_{(m)}\, \mathbb{E}(u u^{\mathrm{T}} | \tilde{W}_\alpha))) = \\
&= n^{-1}\,\mathbb{E}(\mathrm{tr}(\tilde{W}_\alpha^{\mathrm{T}} W_{(m)})) = \\
&= n^{-1} \iota_n^{\mathrm{T}}\,\mathbb{E}(\tilde{W}_\alpha \odot W_{(m)}) \iota_n \,.
\end{aligned} \tag{SM.10}$$

From Equation (SM.6) it follows that:

$$\begin{aligned}
\mathbb{E}(\tilde{W}_\alpha \odot W_{(m)}) &= \mathbb{E}(\tilde{W}_\alpha) \odot W_{(m)} = \\
&= \gamma (n-1)^{-1} \lfloor \alpha m \rfloor \iota_n^{\mathrm{T}} \odot W_{(m)} + (1-\gamma) W_{(\lfloor (1\wedge\alpha) m \rfloor)} = \\
&= \gamma (n-1)^{-1} \Lambda W_{(m)} + (1-\gamma) W_{(\lfloor (1\wedge\alpha) m \rfloor)} \,,
\end{aligned} \tag{SM.11}$$

where $\Lambda \overset{\text{def}}{=} \mathrm{diag}(\lfloor \alpha m \rfloor)$.

Hence, if (SM.11) is substituted in (SM.10), we have:

$$\begin{aligned}
& n^{-1} \iota_n^{\mathrm{T}}\,\mathbb{E}(\tilde{W}_\alpha \odot W_{(m)}) \iota_n = \\
& = \gamma n^{-1} (n-1)^{-1} \lfloor \alpha m \rfloor^{\mathrm{T}} m + (1-\gamma) n^{-1} \iota_n^{\mathrm{T}} \lfloor (1\wedge\alpha) m \rfloor = \\
& \approx \gamma (n-1)^{-1} \left(\alpha (1+\kappa_m^2) \bar{m}^2 - \bar{m}\bar{d}_\alpha\right) + (1-\gamma)\left((1\wedge\alpha)\bar{m} - \bar{d}_{1\wedge\alpha}\right) \,,
\end{aligned}$$

because of (SM.4), and since

$$\begin{aligned}
\iota_n^{\mathrm{T}} \lfloor (1\wedge\alpha) m \rfloor &= (1\wedge\alpha) \iota_n^{\mathrm{T}} m - \iota_n^{\mathrm{T}} \left((1\wedge\alpha) m - \lfloor (1\wedge\alpha) m \rfloor\right) = \\
&= (1\wedge\alpha) n \bar{m} - n \bar{d}_{(1\wedge\alpha)} \,,
\end{aligned} \tag{SM.12a}$$

$$\begin{aligned}
\lfloor \alpha m \rfloor^{\mathrm{T}} m &= \alpha m^{\mathrm{T}} m - \left(\alpha m - \lfloor \alpha m \rfloor\right)^{\mathrm{T}} m = \\
&= \alpha (\bar{m}^2 \kappa_m^2 + \bar{m}^2) - d_\alpha^{\mathrm{T}} m \approx \\
&\approx \alpha (1+\kappa_m^2) \bar{m}^2 - \bar{m}\bar{d}_\alpha \,,
\end{aligned} \tag{SM.12b}$$

according to (SM.7).

Analogously, the variance of the elements of $\tilde{W}_\alpha u$ can be computed as follows:

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}(\tilde{W}_\alpha u)_i^2\right) = n^{-1} \iota_n^{\mathrm{T}}\,\mathbb{E}(\tilde{W}_\alpha \odot \tilde{W}_\alpha) \iota_n \,.$$

Note that $\tilde{W}_\alpha \odot \tilde{W}_\alpha = \tilde{W}_\alpha$, since $\tilde{W}_\alpha$ is binary, hence:

$$\mathbb{E}(\tilde{W}_\alpha \odot \tilde{W}_\alpha) = \mathbb{E}(\tilde{W}_\alpha)$$

and thus, according to Equation (SM.5) and (SM.12a), we have that:

$$\begin{aligned} n^{-1}\iota_n^{\mathrm{T}}\,\mathbb{E}(\tilde{W}_\alpha \odot \tilde{W}_\alpha)\iota_n &= n^{-1}\iota_n^{\mathrm{T}}\,\mathbb{E}(\tilde{W}_\alpha)\iota_n = \\ &= \gamma n^{-1}\iota_n^{\mathrm{T}}\lfloor \alpha m\rfloor + (1-\gamma)n^{-1}\iota_n^{\mathrm{T}}\lfloor \alpha m\rfloor = \\ &= \alpha\bar{m} - \bar{d}_\alpha\,. \end{aligned}$$

Finally, the variance of the elements of $W_{(m)}u$ can be computed as it follows:

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}(W_{(m)}u)_i^2\right) = n^{-1}\iota_n^{\mathrm{T}}(W_{(m)} \odot W_{(m)})\iota_n = \bar{m}\,.$$

It is now possible to determine the correlation coefficient between the elements of $\tilde{W}_\alpha u$ and $W_{(m)}u$ as it follows:

$$\begin{aligned} \operatorname{cor}(\tilde{W}_\alpha u, W_{(m)}u) &= \frac{\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}(\tilde{W}_\alpha u)_i(W_{(m)}u)_i\right)}{\sqrt{\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}(\tilde{W}_\alpha u)_i^2\right)\cdot\mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}(W_{(m)}u)_i^2\right)}} = \\ &= \frac{\gamma\frac{\bar{m}}{n-1}\left[\alpha(1+\kappa_m^2) - \frac{\bar{d}_\alpha}{\bar{m}}\right] + (1-\gamma)\left[(1\wedge\alpha) - \frac{\bar{d}_{1\wedge\alpha}}{\bar{m}}\right]}{\sqrt{\alpha\left(1-\frac{\bar{d}_\alpha}{\alpha\bar{m}}\right)}} = \\ &= \gamma\,\frac{\bar{m}}{n-1}\,(1+\kappa_m^2)\sqrt{\alpha}\,\xi_1 + (1-\gamma)\,\frac{\sqrt{\alpha}}{\max\{1,\alpha\}}\,\xi_2\,, \end{aligned} \tag{SM.13}$$

where

$$\xi_1 \stackrel{\text{def}}{=} \frac{1-\frac{\bar{d}_\alpha}{(1+\kappa_m^2)\alpha\bar{m}}}{\sqrt{1-\frac{\bar{d}_\alpha}{\alpha\bar{m}}}}\,, \qquad \xi_2 \stackrel{\text{def}}{=} \frac{1-\frac{\bar{d}_{1\wedge\alpha}}{(1\wedge\alpha)\bar{m}}}{\sqrt{1-\frac{\bar{d}_\alpha}{\alpha\bar{m}}}}\,. \tag{SM.14}$$

If $\bar{d}_\alpha = 0$ then $\xi_1 = 1$ and $\xi_2 = 1$, and Equation (SM.13) becomes Equation (11). Moreover, if $\kappa_m^2 = 0$, Equation (10) is obtained.

Properties (SM.9) permit one to verify that $\bar{d}_{1\wedge\alpha} = 0$ if $\alpha \geq 1$, hence $\bar{d}_{1\wedge\alpha} = \mathbb{1}_{\{\alpha<1\}}\bar{d}_{1\wedge\alpha}$ (being $\mathbb{1}_{\{\cdot\}}$ the indicator function). Basic algebra manipulations permit the following inequalities to be derived:

$$\begin{aligned} 1 \leq \quad &\xi_1 \quad \leq 1 + \frac{\kappa_m^2}{1+\kappa_m^2}\,(\alpha\bar{m}-1)^{-1}\,, \\ 1 - \mathbb{1}_{\{\alpha<1\}}\,(\alpha\bar{m})^{-1} \leq \quad &\xi_2 \quad \leq 1 + \mathbb{1}_{\{\alpha>1\}}\,(\alpha\bar{m}-1)^{-1}\,, \end{aligned}$$

for $\alpha\bar{m} > 1$, whereas the condition $\alpha\bar{m} \leq 1$ is practically irrelevant. □

4 Proof of Equation (13)

Proof. If the binary matrix $H \in \{0,1\}^{n\times n}$ is defined as:

$$H_{ij} = \mathbb{1}_{\{(W)_{ij}>0\}}\,,$$

pertrurbation (12) can be restated in matrix form as it follows:

$$\tilde{W} = V \odot (Q_n - B) \odot H + W \odot B\,, \tag{SM.15}$$

where $Q_n \overset{\text{def}}{=} \iota_n \iota_n^{\mathrm{T}} - I_n$.

From (SM.15), it follows that:

$$\begin{aligned} \tilde{W} \odot \tilde{W} &= V \odot V \odot (Q_n - B) + W \odot W \odot B\,, \\ \tilde{W} \odot W &= V \odot W \odot (Q_n - B) + W \odot W \odot B\,, \end{aligned}$$

since $(Q_n - B) \odot B = 0$. Thus:

$$\begin{aligned} \mathbb{E}(\tilde{W} \odot \tilde{W}) &= \gamma(1+\kappa_V^2)\mu_V^2 H + (1-\gamma)(1+\kappa_W^2)\mu_W^2 H\,, \\ \mathbb{E}(\tilde{W} \odot W) &= \gamma(1+\kappa_W\kappa_V\rho_{WV})\mu_W\mu_V H + (1-\gamma)(1+\kappa_W^2)\mu_W^2 H\,, \\ \mathbb{E}(W \odot W) &= (1+\kappa_W^2)\mu_W^2 H\,. \end{aligned}$$

Finally we have that:

$$\begin{aligned} \mathrm{cor}(\tilde{W}u, Wu) &= \frac{n^{-1}\iota_n^{\mathrm{T}}\,\mathbb{E}(\tilde{W} \odot W)\iota_n}{\sqrt{n^{-1}\iota_n^{\mathrm{T}}\,\mathbb{E}(W \odot W)\iota_n \cdot n^{-1}\iota_n^{\mathrm{T}}\,\mathbb{E}(\tilde{W} \odot \tilde{W})\iota_n}} = \\ &= \frac{\gamma(1+\kappa_W\kappa_V\rho_{WV})\mu_W\mu_V + (1-\gamma)(1+\kappa_W^2)\mu_W^2}{\sqrt{(1+\kappa_W^2)\mu_W^2\left[\gamma(1+\kappa_V^2)\mu_V^2 + (1-\gamma)(1+\kappa_W^2)\mu_W^2\right]}} = \\ &= \frac{\gamma(1+\kappa_W\kappa_V\rho_{WV})\eta + (1-\gamma)(1+\kappa_W^2)}{\sqrt{(1+\kappa_W^2)\left[\gamma(1+\kappa_V^2)\eta^2 + (1-\gamma)(1+\kappa_W^2)\right]}}\,. \end{aligned}$$

This completes the proof. □

5 Proof of Equations (14) and (15)

5.1 Proof of Equation (15a)

Proof. For the sake of notational convenience, define the following quantities:

$$a_W = 1+\kappa_V^2 \qquad a_V = 1+\kappa_V^2 \qquad a_{WV} = 1+\kappa_W\kappa_V\rho_{WV}\,, \tag{SM.16}$$

and the function g as it follows:

$$g(\gamma, \eta, a_W, a_V) = a_W\left[\gamma\, a_V\eta^2 + (1-\gamma)a_W\right]\,;$$

then note that:

$$\frac{\partial g}{\partial \eta} = 2\gamma\, a_W a_V \eta\,.$$

It is now possible to compute the first derivative of (13) with respect to η as it follows:

$$\begin{aligned} \frac{\partial}{\partial \eta}\,\mathrm{cor}(\tilde{W}_{\bar{m}}u, W_{\bar{m}}u) &= \frac{\partial}{\partial \eta}\left(\frac{\gamma\, a_{WV}\eta + (1-\gamma)a_W}{\sqrt{g(\gamma,\eta,a_W,a_V)}}\right) = \\ &= \frac{\gamma\, a_{WV}\, g(\gamma,\eta,a_W,a_V) - \left[\gamma\, a_{WV}\eta + (1-\gamma)a_W\right]\gamma\, a_W a_V \eta}{\left[g(\gamma,\eta,\kappa_W,\kappa_V)\right]^{3/2}} = \\ &= \frac{\gamma(1-\gamma)\, a_W^2\,(a_{WV} - a_V\eta)}{\left(a_W\left[\gamma\, a_V\eta^2 + (1-\gamma)a_W\right]\right)^{3/2}}\,. \end{aligned}$$

If the previous derivative is set to zero and the equation is solved with respect to η, optimality condition (15a) is found. □

5.2 Proof of Equation (15b)

Proof. Using notation shortcuts defined in (SM.16), it is possible to compute the first derivative of (13) with respect to κ_V as it follows:

$$\frac{\mathrm{d}}{\mathrm{d}\kappa_V}\operatorname{cor}(\tilde{W}_{\bar{m}}u, W_{\bar{m}}u) = \frac{\mathrm{d}}{\mathrm{d}\kappa_V}\left(\frac{\gamma\, a_{WV}\eta + (1-\gamma)a_W}{\sqrt{g(\gamma,\eta,a_W,a_V)}}\right) =$$

$$= \gamma\eta\, a_W\, \frac{\rho_{WV}\kappa_W\left[\gamma\, a_V\eta^2 + (1-\gamma)a_W\right] - \left[\gamma\, a_{WV}\eta + (1-\gamma)a_W\right]\kappa_V\eta}{\left[g(\gamma,\eta,\kappa_W,\kappa_V)\right]^{3/2}} =$$

$$= \gamma\eta(1+\kappa_W^2)\, \frac{\gamma\eta^2(\rho_{WV}\kappa_W - \kappa_V) + (1-\gamma)(1+\kappa_W^2)(\rho_{WV}\kappa_W - \kappa_V\eta)}{\left[g(\gamma,\eta,\kappa_W,\kappa_V)\right]^{3/2}} \,.$$

If the previous derivative is set to zero and the equation is solved with respect to κ_V, optimality condition (15b) is found. □

5.3 Proof of Equation (14)

Proof. Correlation (13) is maximised with respect to η and κ_V if both conditions (15) are satisfied. It follows that if the system of two equations (15) is solved with respect to η and κ_V, solution (14) is found.

This can be easily verified if (15a) is substituted in (13), and the result is maximised with respect to κ_V:

$$\frac{\mathrm{d}}{\mathrm{d}\kappa_V}\left(\frac{\gamma\, a_{WV}\eta + (1-\gamma)a_W}{\sqrt{g(\gamma,\eta,a_W,a_V)}}\right) =$$

$$= \frac{\mathrm{d}}{\mathrm{d}\kappa_V}\left(\frac{\gamma\, a_{WV}^2\, a_V^{-1} + (1-\gamma)a_W}{\sqrt{a_W\left[\gamma\, a_{WV}^2\, a_V^{-1} + (1-\gamma)a_W\right]}}\right) = \frac{\mathrm{d}}{\mathrm{d}\kappa_V}\left(\sqrt{1-\gamma+\gamma\,\frac{a_{WV}^2}{a_W\, a_V}}\right) =$$

$$= \left(1-\gamma+\gamma\,\frac{a_{WV}^2}{a_W\, a_V}\right)^{-1/2} \gamma\,\frac{a_{WV}}{a_W\, a_V^{-2}}\left(\kappa_W\rho_{WV} - \kappa_V\right) \,.$$

If the previous derivative is set to zero and the equation is solved with respect to κ_V, optimality condition (14) is found for κ_V. If $\kappa_V^* = \kappa_W\rho_{WV}$ is substituted in (15a), optimality condition $\eta^* = 1$ in (14) is found. □

References

Horn, R. A. and C. R. Johnson (2013). *Matrix Analysis*. 2nd ed. Cambridge University Press, New York.

LeSage, J. P. and R. K. Pace (2009). *Introduction to Spatial Econometrics*. Chapmann&Hall.