A frequency domain test for isotropy in spatial data models

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Supplementary Material

1 Proof of Theorem 1

Both Points (a) and (b) derive from Parseval's identity (see e.g. Grafakos 2008) which ensures that:

$$\|f - c_0\|_2^2 = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} |c_n|^2 , \qquad \|\tilde{f}_{q,\psi} - f\|_2^2 = \sum_{n=-\infty}^{+\infty} |\tilde{c}_n - c_n|^2 .$$

Point (c). Let $H^{(\alpha)} \colon \mathbb{R} \mapsto \{0,1\}$ be a function of period 2π defined as:

$$H^{(\alpha)}(\omega) \equiv \begin{cases} 1 & \text{if } \exists k \in \mathbb{Z} \colon \omega + 2\pi k \in [0, \alpha) \\ 0 & \text{if } \exists k \in \mathbb{Z} \colon \omega + 2\pi k \in [\alpha, 2\pi) \end{cases}$$

It can be verified that H can be expanded in Fourier series as follows:

$$H^{(\alpha)}(\omega) = \sum_{n=-\infty}^{+\infty} \frac{1 - \mathrm{e}^{-\mathrm{i}n\alpha}}{2\pi \mathrm{i} n} \mathrm{e}^{\mathrm{i}n\omega} ,$$

where the limit for $n \to 0$ is taken as the coefficient of $\mathrm{e}^{\mathrm{i}n\omega}$ at $n = 0$.

The q-ABE approximation $\tilde{f}_{q,\psi}$ of f with reference angle ψ can be restated as follows:

$$\begin{aligned} \tilde{f}_{q,\psi}(\omega) &= \sum_{r=1}^{q} \rho_r H^{(2\pi/q)}(\omega - \psi - 2\pi(r-1)/q) = \\ &= \sum_{n=-\infty}^{+\infty} \frac{\mathrm{i}}{2\pi n} \left[\sum_{r=1}^{q} \mathrm{e}^{-\mathrm{i}n\left(\psi + \frac{2\pi}{q} r\right)} \left(1 - \mathrm{e}^{\mathrm{i}n\frac{2\pi}{q}}\right) \frac{q}{2\pi} \int_{\psi + \frac{2\pi}{q}(r-1)}^{\psi + \frac{2\pi}{q} r} f(s)\mathrm{d}s \right] \mathrm{e}^{\mathrm{i}n\omega} , \end{aligned}$$

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where the last equality results form Equation (10) and some algebraic rearrangements.

Point (d). It follows directly from Equation (SM.3).

Point (e). Equation (13) can be restated as:

$$\tilde{c}_n = \frac{1}{2\pi}\sum_{r=1}^{q}\frac{2\pi}{q}\left(\frac{q}{2\pi}\int_{\theta+\frac{2\pi}{q}(r-1)}^{\theta+\frac{2\pi}{q}r}\mathrm{e}^{-ins}\mathrm{d}s\right)\left(\frac{q}{2\pi}\int_{\theta+\frac{2\pi}{q}(r-1)}^{\theta+\frac{2\pi}{q}r}f(s)\mathrm{d}s\right).$$

The previous equation is bounded by the lower and the upper Darboux sums (see e.g. Apostol 1974) of the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}n\omega}f(\omega)\,\mathrm{d}\omega\,, \tag{SM.1}$$

where the integration interval $[0, 2\pi]$ is partitioned into q equally-wide subintervals; it follows that Equation (13) converges to Equation (SM.1) as $q \to \infty$. Since Equation (SM.1) is the n-th Fourier coefficient c_n of the function f, Point (e) is proved.

Point (f). It follows from Equation (SM.3). This completes the proof.

Point (g). A constructive proof is provided. Given (q_0, ψ_0), the sequence $\{(2^n q_0, \psi_0)\}$ for $n \geq 0$ implies that $\tilde{f}_{q_n,\psi_n} \xrightarrow{\mathcal{L}^2} f$ as $n \to \infty$. This can be proved as follows. (For the sake of notational simplicity, we define $f_n \equiv \tilde{f}_{q_n,\psi_n}$.)

As noted previously, f_n is a step function. It follows that, for any n, we have:

$$\|f_n - f\|_2^2 = \sum_{r=1}^{q_n}\int_{I_r}(f(\omega)-\rho_r)^2\,\mathrm{d}\omega\,,$$

where ρ_r is defined in (10).

When the approximation f_{n+1} is considered, for any $r = 1,\ldots,q_n$, the interval $I_r = [\psi_0 + \frac{2\pi}{q_n}(r-1),\ \psi_0 + \frac{2\pi}{q_n}r)$ is split into two intervals $I'_r = [\psi_0 + \frac{2\pi}{q_n}(r-1),\ \psi_0 + \frac{2\pi}{q_n}(r-\frac{1}{2}))$ and $I''_r = [\psi_0 + \frac{2\pi}{q_n}(r-\frac{1}{2}),\ \psi_0 + \frac{2\pi}{q_n}r)$. Hence the norm $\|f_n - f\|_2^2$ over I_r may be restated as follows:

$$\begin{aligned}
&\int_{I_r}(f(\omega)-f_n(\omega))^2\,\mathrm{d}\omega = \int_{I_r}(f(\omega)-\rho_r)^2\,\mathrm{d}\omega = \\
&= \int_{I'_r}(f(\omega)-\rho'_r)^2\,\mathrm{d}\omega + \int_{I''_r}(f(\omega)-\rho''_r)^2\,\mathrm{d}\omega + \frac{\pi}{q_n}(\rho'_r-\rho_r)^2 + \frac{\pi}{q_n}(\rho''_r-\rho_r)^2 = \\
&= \int_{I_r}(f(\omega)-f_{n+1}(\omega))^2\,\mathrm{d}\omega + \frac{\pi}{q_n}(\rho'_r-\rho_r)^2 + \frac{\pi}{q_n}(\rho''_r-\rho_r)^2\,,
\end{aligned}$$

where $\rho'_r = \frac{q}{\pi}\int_{I'_r}f(s)\,\mathrm{d}s$, and $\rho''_r = \frac{q}{\pi}\int_{I''_r}f(s)\,\mathrm{d}s$.

It follows that $\|f_{n+1} - f\|_2^2 \leq \|f_n - f\|_2^2$, with $\|f_{n+1} - f\|_2^2 = \|f_n - f\|_2^2$ only if $\rho = \rho' = \rho''$.

By induction, we can conclude that $\|f_m - f\|_2^2 \leq \|f_n - f\|_2^2$ for any $m > n$, and $\|f_m - f\|_2^2 = \|f_n - f\|_2^2$ only if:

$$\frac{q_n}{2\pi}\int_{I_r}f(\omega)\,\mathrm{d}\omega = \frac{q_m}{2\pi}\int_{\psi_0+\frac{2\pi}{q_m}(s-1)}^{\psi_0+\frac{2\pi}{q_m}s}f(\omega)\,\mathrm{d}\omega\,,$$

for any $r \in \{1, \ldots, q_n\}$ and any $s \in \{1, \ldots, q_m\}$.

If $\|f_n - f\|_2^2 > 0$ for some $n \geq 0$, there exists an interval $A = [\psi_0 + \frac{2\pi}{q_m}(s - 1),\ \psi_0 + \frac{2\pi}{q_m}s) \subseteq [0, 2\pi)$ with $m > n$ and $s \in \{1, \ldots, q_m\}$ such that:

$$\int_A (f(\omega) - f_n(\omega))\,\mathrm{d}\omega \neq 0\,,$$

and thus $\|f_m - f\|_2^2 < \|f_n - f\|_2^2$, since f_m is the least squared constant function of f over A.

Point (h). Note that Equation (13) can be restated as follows:

$$\begin{aligned}
\tilde{c}_n &= \frac{\mathrm{i}}{2\pi n} \sum_{r=1}^{q} \left[\mathrm{e}^{-\mathrm{i}n\left(\psi + \frac{2\pi}{q} r\right)} \left(1 - \mathrm{e}^{\mathrm{i}n\frac{2\pi}{q}}\right) \frac{q}{2\pi} \int_{\psi + \frac{2\pi}{q}(r-1)}^{\psi + \frac{2\pi}{q} r} f(s)\,\mathrm{d}s \right] = \\
&= \frac{\mathrm{i}}{2\pi n} \sum_{r=1}^{q} \left[\mathrm{e}^{-\mathrm{i}n\left(\psi + \frac{2\pi}{q} r\right)} \left[1 - e^{2\pi\mathrm{i}\left(\frac{n}{q} - \lfloor \frac{n}{q} \rfloor\right)}\right] \frac{q}{2\pi} \int_{\psi + \frac{2\pi}{q}(r-1)}^{\psi + \frac{2\pi}{q} r} f(s)\,\mathrm{d}s \right] = \\
&= \frac{1}{2\pi} \left[\frac{1 - e^{2\pi\mathrm{i}\left(\frac{n}{q} - \lfloor \frac{n}{q} \rfloor\right)}}{-2\pi\mathrm{i}\frac{n}{q}} \right] \mathrm{e}^{-\mathrm{i}\frac{n}{q} q\psi} \sum_{r=1}^{q} \left[\mathrm{e}^{-2\pi\mathrm{i}\frac{n}{q} r} \int_{\psi + \frac{2\pi}{q}(r-1)}^{\psi + \frac{2\pi}{q} r} f(s)\,\mathrm{d}s \right],
\end{aligned} \tag{SM.2}$$

being $\lfloor \cdot \rfloor$ the floor function.

The modulus of Equation (SM.2) is:

$$\begin{aligned}
|\tilde{c}_n| &= \frac{1}{2\pi} \left| \frac{1 - e^{2\pi\mathrm{i}\left(\frac{n}{q} - \lfloor \frac{n}{q} \rfloor\right)}}{-2\pi\mathrm{i}\frac{n}{q}} \right| \cdot \left| \sum_{r=1}^{q} \left[\mathrm{e}^{-2\pi\mathrm{i}\frac{n}{q} r} \int_{\psi + \frac{2\pi}{q}(r-1)}^{\psi + \frac{2\pi}{q} r} f(s)\,\mathrm{d}s \right] \right| \leq \\
&\leq \frac{1}{2\pi} \left| \frac{1 - e^{2\pi\mathrm{i}\left(\frac{n}{q} - \lfloor \frac{n}{q} \rfloor\right)}}{-2\pi\mathrm{i}\frac{n}{q}} \right| \cdot \left| \frac{1 - \mathrm{e}^{-2\pi\frac{n}{q} q}}{1 - \mathrm{e}^{-2\pi\frac{n}{q}}} \right| \cdot \|f\|_\infty = \left| \frac{1 - e^{2\pi\mathrm{i}\left(\frac{n}{q} - \lfloor \frac{n}{q} \rfloor\right) q}}{n/q} \right| \cdot \frac{\|f\|_\infty}{4\pi^2} = \\
&= \left| \frac{n}{q} \right|^{-1} \cdot \sqrt{1 - \cos\left(2\pi\left(\frac{n}{q} - \left\lfloor \frac{n}{q} \right\rfloor\right) q\right)} \cdot \frac{\|f\|_\infty}{4\pi^2}\,,
\end{aligned} \tag{SM.3}$$

it follows that $\tilde{c}_n$ lies in a circle centered on the origin of the complex plane and having radius (SM.3).

For any $k \in \mathbb{Z}^*$, if $n = kq$ the radius (SM.3) is zero, hence ψ does not affect $|\tilde{c}_n - c_n|$. If $k = 0$, then $n = 0$ and $\tilde{c}_0 = c_0$ whatever is the value of ψ.

The effect of ψ on $|\tilde{c}_n - c_n|$ depends mainly on f, anyway, the bound in (SM.3) represents a restriction on $\tilde{c}_n$ that may be binding or not. However, the closer is the value of (SM.3) to zero, the stronger is the restriciton in spite of ψ; on the other hand, the larger is (SM.3), the greater is the possibility to minimise $|\tilde{c}_n - c_n|$ with respect to ψ.

In order to prove that the radius (SM.3) is maximum over $[kq,\ (k+1)q]$ approximately at $n = (k + 1/2)q$, define $H(x) \equiv |x|^{-1}$ and define $G(x) \propto \sqrt{1 - \cos(2\pi(x - \lfloor x \rfloor))}$ such that $H(n/q)\,G(n/q)$ equals (SM.3). From calculus it follows that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(H(x)\,G(x)\right) = -\frac{G(x)}{x^2} + \frac{1}{x}\frac{\mathrm{d}G(x)}{\mathrm{d}x}\,.$$

It follows from the previous equation that the first order condition for a local stationary point of $H(x)\,G(x)$ converges to the first order condition of $G(x)$ as x

gets large in absolute value. As a consequence, for large x the stationary points of $H(x)\,G(x)$ can be approximated by those of $G(x)$. In particular, $G(n/q)$ is maximised for $n = (k+1/2)q$, for any $k \in \mathbb{Z}$. The maximum distance ($k = 1$) between the true and approximated maximising point is approximately 0.07.

The descending effect of ψ on $|\tilde{c}_n - c_n|$ can be appreciated by noticing that (SM.3) converges to zero as $n/q \to \infty$.

Point (i). The proof is immediate, as each ρ_r results from the integration of the n-th harmonics over either its positive or negative values. This is guaranteed by the fact that there are two ρs for each cycle of the harmonics ($q = 2n$), whereas $\psi = \mathrm{Arg}(c_n)$ keep the waves in-phase. □

2 Results of Simulation Study

The present section includes the following tables and figures:

- **Table SM.1** – Empirical power of the FD-test on model (19) for various values of ρ_1 and ρ_2.
- **Table SM.2** – Empirical power of 2-ABE, 4-ABE and FD test on model (20) for various values of α, ψ and ν, and for $\sigma = 1$.
- **Table SM.3** – Empirical power of 2-ABE, 4-ABE and FD test on model (20) for various values of α, ψ and ν, and for $\sigma = 1/2$.
- **Table SM.4** – Empirical power of 2-ABE, 4-ABE and FD test on model (20) for various values of α, ψ and ν, and for $\sigma = 1/4$.

References

Apostol, T. M. (1974). *Mathematical Analysis.* 2nd ed. Prentice Hall.
Grafakos, L. (2008). *Classical Fourier Analysis.* 2nd ed. Springer.

	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10	0.00
−0.90	–	–	–	–	–	–	–	–	–	1.000
−0.80	–	–	–	–	–	–	–	–	1.000	1.000
−0.70	–	–	–	–	–	–	–	0.993	1.000	1.000
−0.60	–	–	–	–	–	–	0.766	0.943	0.995	1.000
−0.50	–	–	–	–	–	0.157	0.436	0.725	0.947	0.990
−0.40	–	–	–	–	0.154	0.064	0.126	0.404	0.726	0.939
−0.30	–	–	–	0.773	0.429	0.139	0.054	0.153	0.428	0.754
−0.20	–	–	0.992	0.945	0.752	0.422	0.141	0.042	0.140	0.417
−0.10	–	1.000	1.000	0.993	0.950	0.751	0.414	0.139	0.051	0.148
0.00	1.000	0.999	1.000	1.000	0.991	0.933	0.751	0.392	0.144	0.035
0.10	–	1.000	1.000	1.000	1.000	0.991	0.913	0.727	0.402	0.142
0.20	–	–	1.000	1.000	1.000	1.000	0.997	0.943	0.764	0.412
0.30	–	–	–	1.000	1.000	1.000	0.999	0.988	0.927	0.750
0.40	–	–	–	–	1.000	1.000	1.000	1.000	0.995	0.943
0.50	–	–	–	–	–	1.000	1.000	1.000	1.000	0.995
0.60	–	–	–	–	–	–	1.000	1.000	1.000	1.000
0.70	–	–	–	–	–	–	–	1.000	1.000	1.000
0.80	–	–	–	–	–	–	–	–	1.000	1.000
0.90	–	–	–	–	–	–	–	–	–	1.000

	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
−0.90	–	–	–	–	–	–	–	–	–
−0.80	1.000	–	–	–	–	–	–	–	–
−0.70	1.000	1.000	–	–	–	–	–	–	–
−0.60	1.000	1.000	1.000	–	–	–	–	–	–
−0.50	1.000	1.000	1.000	1.000	–	–	–	–	–
−0.40	0.987	1.000	1.000	1.000	1.000	–	–	–	–
−0.30	0.943	0.992	1.000	1.000	1.000	1.000	–	–	–
−0.20	0.748	0.921	0.989	0.999	1.000	1.000	1.000	–	–
−0.10	0.400	0.725	0.938	0.995	1.000	1.000	1.000	1.000	–
0.00	0.113	0.408	0.732	0.936	0.992	1.000	1.000	1.000	1.000
0.10	0.060	0.148	0.423	0.748	0.941	0.995	1.000	1.000	–
0.20	0.127	0.045	0.143	0.436	0.740	0.958	0.991	–	–
0.30	0.396	0.152	0.057	0.139	0.403	0.739	–	–	–
0.40	0.746	0.416	0.162	0.052	0.149	–	–	–	–
0.50	0.935	0.759	0.429	0.131	–	–	–	–	–
0.60	0.991	0.939	0.795	–	–	–	–	–	–
0.70	1.000	0.995	–	–	–	–	–	–	–
0.80	1.000	–	–	–	–	–	–	–	–
0.90	–	–	–	–	–	–	–	–	–

Table SM.1: Empirical evaluation of the power of the FD-test on model (19) for various values of ρ_1 (columns) and ρ_2 (rows). The evaluation of the empirical power is based on 1000 simulations for each couple (ρ_1, ρ_2). The coefficient restriction has been tested by means of the likelihood ratio test with a 5% significance level.

α	ABE ($q=2$)				ABE ($q=4$)			FD	FD
	$\psi=0$	$\psi=\pi/6$	$\psi=\pi/3$	$\psi=\pi/2$	$\psi=0$	$\psi=\pi/6$	$\psi=\pi/3$	$\nu=2$	$\nu=3$
−1.0	0.430	0.496	0.318	0.074	0.310	0.764	0.750	0.854	0.862
−0.9	0.332	0.420	0.268	0.058	0.236	0.668	0.662	0.778	0.750
−0.8	0.292	0.318	0.218	0.052	0.190	0.544	0.574	0.654	0.612
−0.7	0.244	0.300	0.204	0.062	0.162	0.446	0.424	0.526	0.512
−0.6	0.200	0.194	0.134	0.050	0.122	0.316	0.354	0.410	0.388
−0.5	0.150	0.174	0.122	0.058	0.114	0.232	0.226	0.328	0.274
−0.4	0.104	0.120	0.110	0.042	0.078	0.152	0.154	0.180	0.162
−0.3	0.076	0.100	0.078	0.056	0.074	0.140	0.108	0.126	0.132
−0.2	0.062	0.072	0.094	0.066	0.066	0.092	0.078	0.088	0.098
−0.1	0.056	0.048	0.042	0.038	0.044	0.058	0.058	0.066	0.054
0.0	0.054	0.050	0.046	0.048	0.046	0.066	0.044	0.046	0.070
0.1	0.056	0.052	0.050	0.058	0.064	0.068	0.044	0.060	0.064
0.2	0.062	0.082	0.060	0.040	0.060	0.080	0.078	0.070	0.078
0.3	0.094	0.090	0.074	0.046	0.064	0.132	0.132	0.126	0.132
0.4	0.100	0.134	0.092	0.062	0.094	0.156	0.154	0.180	0.166
0.5	0.172	0.174	0.116	0.044	0.110	0.282	0.278	0.308	0.298
0.6	0.182	0.210	0.126	0.064	0.156	0.330	0.380	0.446	0.432
0.7	0.252	0.286	0.162	0.060	0.182	0.484	0.492	0.594	0.560
0.8	0.314	0.322	0.204	0.054	0.220	0.594	0.588	0.754	0.694

Table SM.2: Empirical evaluation of the power of the 2-ABE, the 4-ABE and the FD-test for model (20) based on the function (17) with $\kappa = \pi/2$ and several values of α. The ABE tests have been performed for various values of the reference direction ψ, whilst the FD-test uses two ($\nu = 2$) and three ($\nu = 3$) harmonics. For each value of α, 500 models (20) with $n = 400$ and $\sigma = 1$ have been simulated. The coefficient restrictions of the isotropy tests have been tested by means of the likelihood ratio test at the 5% significance level.

α	ABE ($q=2$)				ABE ($q=4$)			FD	FD
	$\psi=0$	$\psi=\pi/6$	$\psi=\pi/3$	$\psi=\pi/2$	$\psi=0$	$\psi=\pi/6$	$\psi=\pi/3$	$\nu=2$	$\nu=3$
−1.0	0.986	0.998	0.938	0.066	0.960	1.000	1.000	1.000	1.000
−0.9	0.960	0.990	0.892	0.082	0.920	1.000	1.000	1.000	1.000
−0.8	0.926	0.972	0.834	0.064	0.880	0.996	0.998	1.000	1.000
−0.7	0.854	0.902	0.696	0.070	0.732	0.976	0.982	0.992	0.992
−0.6	0.768	0.828	0.580	0.050	0.614	0.934	0.922	0.968	0.956
−0.5	0.572	0.634	0.434	0.052	0.434	0.764	0.800	0.858	0.842
−0.4	0.424	0.540	0.294	0.052	0.298	0.614	0.600	0.682	0.642
−0.3	0.274	0.266	0.216	0.040	0.184	0.316	0.366	0.378	0.368
−0.2	0.156	0.176	0.124	0.046	0.120	0.164	0.164	0.218	0.210
−0.1	0.074	0.072	0.054	0.046	0.066	0.080	0.058	0.066	0.060
0.0	0.040	0.040	0.068	0.058	0.052	0.048	0.054	0.050	0.066
0.1	0.080	0.084	0.068	0.034	0.062	0.082	0.070	0.084	0.094
0.2	0.148	0.174	0.114	0.046	0.100	0.178	0.184	0.198	0.198
0.3	0.270	0.312	0.204	0.074	0.232	0.374	0.418	0.448	0.424
0.4	0.466	0.520	0.358	0.056	0.366	0.640	0.658	0.730	0.718
0.5	0.624	0.724	0.482	0.060	0.488	0.812	0.860	0.906	0.888
0.6	0.796	0.866	0.664	0.076	0.684	0.956	0.962	0.992	0.982
0.7	0.902	0.952	0.794	0.088	0.826	0.990	0.994	1.000	1.000
0.8	0.982	0.974	0.870	0.084	0.932	1.000	1.000	1.000	1.000

Table SM.3: Empirical evaluation of the power of the 2-ABE, the 4-ABE and the FD-test for model (20) based on the function (17) with $\kappa = \pi/2$ and several values of α. The ABE tests have been performed for various values of the reference direction ψ, whilst the FD-test uses two ($\nu = 2$) and three ($\nu = 3$) harmonics. For each value of α, 500 models (20) with $n = 400$ and $\sigma = 1/2$ have been simulated. The coefficient restrictions of the isotropy tests have been tested by means of the likelihood ratio test at the 5% significance level.

α	ABE ($q=2$)				ABE ($q=4$)			FD	FD
	$\psi=0$	$\psi=\pi/6$	$\psi=\pi/3$	$\psi=\pi/2$	$\psi=0$	$\psi=\pi/6$	$\psi=\pi/3$	$\nu=2$	$\nu=3$
−1.0	1.000	1.000	1.000	0.018	1.000	1.000	1.000	1.000	1.000
−0.9	1.000	1.000	1.000	0.018	1.000	1.000	1.000	1.000	1.000
−0.8	1.000	1.000	1.000	0.020	1.000	1.000	1.000	1.000	1.000
−0.7	1.000	1.000	0.986	0.018	1.000	1.000	1.000	1.000	1.000
−0.6	0.996	0.998	0.982	0.034	0.986	1.000	1.000	1.000	1.000
−0.5	0.976	0.992	0.916	0.046	0.928	1.000	1.000	1.000	1.000
−0.4	0.902	0.936	0.780	0.038	0.778	0.990	0.996	0.998	1.000
−0.3	0.740	0.808	0.592	0.042	0.572	0.908	0.920	0.962	0.962
−0.2	0.378	0.466	0.284	0.044	0.270	0.576	0.544	0.622	0.598
−0.1	0.106	0.134	0.104	0.062	0.098	0.172	0.150	0.186	0.162
0.0	0.062	0.046	0.060	0.048	0.064	0.052	0.070	0.052	0.054
0.1	0.150	0.154	0.098	0.038	0.086	0.180	0.174	0.198	0.184
0.2	0.442	0.450	0.290	0.046	0.300	0.590	0.582	0.676	0.660
0.3	0.730	0.778	0.520	0.064	0.560	0.926	0.922	0.958	0.954
0.4	0.934	0.958	0.798	0.032	0.878	0.992	1.000	1.000	1.000
0.5	0.982	0.994	0.904	0.036	0.950	1.000	1.000	1.000	1.000
0.6	1.000	0.998	0.978	0.036	0.992	1.000	1.000	1.000	1.000
0.7	1.000	1.000	0.996	0.022	1.000	1.000	1.000	1.000	1.000
0.8	1.000	1.000	0.948	0.034	1.000	1.000	1.000	1.000	1.000

Table SM.4: Empirical evaluation of the power of the 2-ABE, the 4-ABE and the FD-test for model (20) based on the function (17) with $\kappa = \pi/2$ and several values of α. The ABE tests have been performed for various values of the reference direction ψ, whilst the FD-test uses two ($\nu = 2$) and three ($\nu = 3$) harmonics. For each value of α, 500 models (20) with $n = 400$ and $\sigma = 1/4$ have been simulated. The coefficient restrictions of the isotropy tests have been tested by means of the likelihood ratio test at the 5% significance level.